Superalgebraic methods in the classical theory of representations.
Capelli's identity, the Koszul map
and the center of the enveloping algebra $\mathbf{U}(gl(n))$.

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1   Introduction

The theme of the superalgebraic extension of many theories both in Geometry and Algebra has noble origins that lay in Physics and has been at the core of a wide range of works by several prominent mathematicians during the last three decades (see, e.g. [18]). We are not concerned with the general theory here, but we limit ourselves to show how the use of superalgebraic methods sheds new light on some classical themes of representation theory and it leads to significant simplifications of traditional proofs.

In this paper we essentially deal with Capelli's identity[1] and with the center of the enveloping algebra of the general linear Lie algebra $gl(n)$.

Capelli's identity is the cornerstone of the classical theory of algebraic invariants (see, e.g [33], [26]). Distinguished contemporary authors referred to this identity as "mysterious" (see, e.g. [2], [20]) , and it still provides a quite active research field. The main mathematical object in Capelli's identity is a "deteminantal" operator that results (in modern language) in a central element of the enveloping algebra $\mathbf{U}(gl(n))$ of the general Lie algebra $gl(n)$. The generalization of this operator led Capelli to study the mathematical structure that we nowadays call the *center of* $\mathbf{U}(gl(n))$, to prove that it is a polynomial algebra and to explicitly describing a family of free algebraic generators ([12], [13], 1893).

By way of elementary motivation, we show that the proof of Capelli's identity can be reduced to a straightforward computation just by using a touch of superalgebraic notions.

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[1] The celebrated identity that now bears his name was proved in 1887 [10], when he held the chair of Algebra at the University of Naples. Alfredo Capelli graduated from the University of Rome in 1877 and then continued to develop his mathematical skills working as Felice Casorati's assistant at the University of Pavia. Casorati corresponded regularly with Weierstrass and had developed a strong link between Italian and German mathematicians. Capelli spent time at the University of Berlin where he was influenced by Karl Weierstrass and Leopold Kronecker.

This point of view is extended to the study of the enveloping algebra $\mathbf{U}(gl(n))$. The notions of *determinantal and permanental Capelli bitableaux* provide two relevant classes of bases that arise from *Straightening Laws* (see. e.g. [6], [17], [19], [26]).

In the final section, we submit new results on the center of $\mathbf{U}(gl(n))$. These results - which cannot even be expressed without appealing to the superalgebraic notation - allows a variety of classical and recent results to be almost trivially proved and be put under one roof.

The crucial methodological tool of our approach to Capelli's identity and to the center of the enveloping algebra $\mathbf{U}(gl(n))$ is the *superalgebraic method of virtual variables*, which is, in turn, an extension of Capelli's method of *variabili ausilarie*. Capelli introduced the method of "variabili ausiliarie" in order to manage symmetrizer operators in terms of polarization operators, to simplify the study of some skew-symmetrizer operators (namely, the famous central Capelli operator) and developed this idea in a systematic way in his beautiful treatise [14]. As a matter of fact, Hermann Weyl's sketchy proof of Capelli's identity ([33], page 39 ff.) must be regarded as an application of the method of "variabili ausiliarie" in disguise (we refer the reader to the footnote to Theorem 2.1 for further details).

Unfortunately, Capelli's idea was well suited to treat symmetrization, but it did not work in the same efficient way while dealing with skew-symmetrization.

We had to wait the introduction of the notion of *superalgebras* to have the right conceptual framework to treat symmetry and skew-symmetry in one and the same way. To the best of our knowledge, the first mathematician who intuited the connection between Capelli's idea and superalgebras was Koszul in 1981 [23]; Koszul proved that the classical determinantal Capelli operator can be rewritten - in a much simpler way - by adding to the symbols to be dealt with an extra auxiliary symbol that obeys to different commutation relations.

The *supersymmetric method of virtual variables* was developed in its full extent and generality (for the general linear Lie superalgebras $gl(m|n)$ - in the notation of [22]) in the series of notes [4], [5], [6], [7] by Teolis and the present author.

This note is organized as follows. In section 2, we summarize some elementary material required to define the classical Capelli operator and to state Capelli's identity.

In section 3, we introduce the minimum of superalgebraic concepts and notations required to express the Capelli operator in a compact way and to provide a few lines proof of classical Capelli's identity.

In Section 4, we provide a systematic treatment of the method of virtual variables. The starting point of the method is to introduce new symbols, called *virtual variables* ("variabili ausiliarie" in the language of Capelli [14]), which may have different signature (parity, $\mathbb{Z}_2-$degree) than the signature of the symbols to be dealt with. This leads to a Lie superalgebra $gl(m_1|m_2 + n)$, where m_1 is the number of positive virtual symbols and m_2 is the number of negative virtual symbols (informally, we assume m_1, m_2 "sufficiently large").

The main technical device of the method is the notion of *virtual algebra* $Virt(m_1 + m_2, n)$ as a subalgebra of the enveloping algebra $\mathbf{U}(gl(m_1|m_2 + n))$. The virtual algebra $Virt(m_1 + m_2, n)$ provides an effective method to define special elements and to derive deep identities in $\mathbf{U}(gl(n))$.

Specifically, $\mathbf{U}(gl(n))$ is the image of $Virt(m_1 + m_2, n)$ under an algebra homomorphism - the Capelli epimorphism - whose kernel is the linear span of a set of monomials characterized by a simple combinatorial property; Capelli rows and bitableaux, while having no simple expressions in $\mathbf{U}(gl(n))$, admit very simple preimages in $Virt(m_1 + m_2, n)$ and, hence, all the calculations are carried on in the virtual algebra, modulo the kernel of the Capelli epimorphism.

In section 5, the sets of *determinantal* and *permanental Capelli bitableaux* are introduced; these elements of $\mathbf{U}(gl(n))$ could not be defined, nor imagined without recourse to the method of virtual variables.

Capelli bitableaux give rise to *straightening laws* for the enveloping algebra [6] that are in all respect similar to the ordinary ones for the classical (symmetric) algebra of algebraic forms $\mathbb{C}[M_{n,n}] \cong Sym[gl(n)]$ (see, e.g. [17], [16], [15], [19]). We developed this connection between $Sym[gl(n)]$ and $\mathbf{U}(gl(n))$ by introducing a linear isomorphism $\mathfrak{T} : \mathbb{C}[M_{n,n}] \cong Sym[gl(n)] \to \mathbf{U}(gl(n))$, called the *bitableau correspondence* [6]; the isomorphism $\mathfrak{T}$ maps each bitableau of $\mathbb{C}[M_{n,n}]$ to the Capelli bitableaux of $\mathbf{U}(gl(n))$ parametrized by the same pair of Young tableaux, both in the determinantal and in the permanental cases. The sets of semistandard determinantal and co-semistandard permanental bitableaux yield two remarkable bases of $\mathbf{U}(gl(n))$; furthermore, the map $\mathfrak{T}$ turns out to be the inverse of the *Koszul map* $\mathfrak{K}$ ([23], [7]).

The final section is devoted to the study of the center $\mathfrak{Z}(\mathbf{U}(gl(n)))$ of the enveloping algebra $\mathbf{U}(gl(n))$. The new concept of *rectangular Capelli-Deruyts tableau* is introduced and a new result (Theorem 6.6) is presented.

Due to their virtual presentation, rectangular Capelli-Deruyts tableaux are almost immediately recognized to be central elements in $\mathbf{U}(gl(n))$ and, by Theorem 6.6, they can be expanded in terms of the central generators first discovered by Capelli in 1893 ([12], [13]).

By combining this result with the beautiful description of the Harish-Chandra isomorphism in the terms of *shifted symmetric polynomials* (Okounkov[2] and Olshanski [25]), a variety of results (see. e.g. [9], [12], [13], [20], [21], [24], [25], [26], [29], [30]) on free sets of generators of the polynomial algebra $\mathfrak{Z}(\mathbf{U}(gl(n)))$ is deduced.

2   The classical Capelli identity

In this section, we essentially refer to [26].

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[2]Fields Medalist 2006

The *algebra of algebraic forms in n vector variables of dimension d* is the polynomial algebra in $n \times d$ variables:

$$\mathbb{C}[M_{n,d}] := \mathbb{C}[x_{ij}]_{i=1,\ldots,n;j=1,\ldots,d}$$

where $M_{n,d}$ represents the matrix with n rows and d columns with "generic" entries x_{ij}:

$$M_{n,d} = [x_{ij}]_{i=1,\ldots,n;j=1,\ldots,d} = \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ x_{21} & \cdots & x_{2d} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nd} \end{bmatrix}.$$

We can interpret x_{ij}, with a slight abuse of notation, as the *multiplication operator*:

$$x_{ij} : \mathbb{C}[M_{n,d}] \to \mathbb{C}[M_{n,d}], \quad x_{ij}(f) = x_{ij} \cdot f, \ \forall f \in \mathbb{C}[M_{n,d}].$$

The *Weyl algebra*

$$\mathbf{W}_{n,d} := \mathbb{C}[x_{ij}, \frac{\partial}{\partial x_{ij}}] \subseteq \mathrm{End}_{\mathbb{C}}[\mathbb{C}[M_{n,d}]]$$

is the subalgebra of $\mathrm{End}_{\mathbb{C}}[\mathbb{C}[M_{n,d}]]$ generated by the multiplication operators x_{ij} and the partial derivative operators $\frac{\partial}{\partial x_{ij}}$.

This is not a commutative algebra:

$$\left[x_{ij}, \frac{\partial}{\partial x_{hk}} \right] = \delta_{ih}\delta_{jk}\mathtt{I},$$

where $\mathtt{I}$ denotes the identity operator.

In $\mathbf{W}_{n,d}$, we consider the *polarization operator*:

$$D_{x_h, x_k} = \sum_{j=1}^{d} x_{hj} \frac{\partial}{\partial x_{kj}}.$$

The operator D_{x_h, x_k} is characterized as the unique derivation of the algebra $\mathbb{C}[M_{n,d}]$ such that

$$D_{x_h, x_k}(x_{ij}) = \delta_{ki} \ x_{hj} \qquad \forall j = 1, \ldots, d.$$

The subalgebra (with 1) of $\mathbf{W}_{n,d}$ generated by the polarization operators is denoted by $\mathbf{P}_{n,d}$ and called the *algebra of polarizations*.

The following identity holds:

$$[D_{x_i, x_j}, D_{x_h, x_k}] = \delta_{jh} D_{x_i, x_k} - \delta_{ik} D_{x_h, x_j}$$

One may recognize the commutation relations of elementary matrices, so we have a representation $\rho : \mathbf{U}(gl(n)) \longrightarrow \mathbf{W}_{n,d}$. If $n \leq d$ we have an injection, so we obtain an isomorphism:

$$\rho(\mathbf{U}(gl(n)) \cong \mathbf{P}_{n,d}$$

Given a square matrix $\mathbf{M_n} = [a_{ij}]_{i,j=1,\ldots,n}$ with entries a_{ij} in a noncommutative algebra, we will consider its *column determinant*

$$\mathbf{cdet}(\mathbf{M_n}) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{|\sigma|} a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}.$$

The *classical Capelli operator* $H_{n,d}$ is the element in $\mathbf{P}_{n,d} \subset \mathbf{W}_{n,d}$ given by the column determinant:

$$H_{n,d} = \mathbf{cdet} \begin{pmatrix} D_{x_1,x_1} + (n-1)\mathtt{I} & D_{x_1,x_2} & \cdots & D_{x_1,x_n} \\ D_{x_2,x_1} & D_{x_2,x_2} + (n-2)\mathtt{I} & \cdots & D_{x_2,x_n} \\ \vdots & & & \vdots \\ D_{x_n,x_1} & D_{x_n,x_2} & \cdots & D_{x_n,x_n} \end{pmatrix} =$$

$$= \mathbf{cdet} \left(D_{x_i,x_j} + \delta_{ij}(n-i)\mathtt{I} \right)_{i,j=1,2,\ldots,n}.$$

The operator $H_{n,d}$ determines a unique element $\mathbf{H}_n$ in $\mathbf{U}(gl(n))$ (with $n \leq d$) which is central (see, e.g. [30]).

The *Cayley operator* Ω_n (see, e.g. [32], [33]):

$$\Omega_n = \mathbf{det} \left(\frac{\partial}{\partial x_{ij}} \right)_{i,j=1,2,\ldots,n}.$$

Notice that Ω_n is computed as a "true" determinant, since its entries are commuting operators.

The *bracket*[3] is the element

$$[x_1,\ldots,x_n] = \mathbf{det} \, [x_{ij}]_{i,j=1,\ldots,n} \in \mathbb{C}[M_{n,d}] \tag{1}$$

We are ready to state the *Capelli's identity.*

Theorem 2.1. *(Capelli, [10] 1887)*

$$H_{n,d}(f) = \begin{cases} 0 & \text{if } n > d \\ [x_1,\ldots,x_n]\Omega_n(f) & \text{if } n = d, \end{cases} \quad \forall f \in \mathbb{C}[M_{n,d}].$$

The case $n = d$ is called the *special identity* ([33]). The special identity[4] may be expressed as the following identity in the Weyl algebra:

$$H_{n,n} = [x_1,\ldots,x_n]\Omega_n.$$

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[3]This notation and terminology is due to Cayley.

[4]As we announced in the Introduction, we formalize Weyl's cumbersome argument ([33], page 39) in the special case $n = d = 2$. Add to the vector symbols $x_1 = (x_{11}, x_{12})$ and

3 The superalgebraic method of virtual variables for Capelli's identities

For the sake of readability, from now on, we will write $(x_i|j)$ in place of x_{ij}.

Besides the (commuting) variables $(x_i|j)$, $i = 1, \ldots, n$, $j = 1, \ldots, d$, we consider a new series of variables $(\alpha|j)$, $j = 1, \ldots, d$ and we assign to them parity equal to one, while we assume the parity of $(x_i|j)$ to be zero:

$$|(x_i|j)| = 0 \in \mathbb{Z}_2, \qquad |(\alpha|j)| = 1 \in \mathbb{Z}_2.$$

So our variables are now $\mathbb{Z}_2$-graded. The elements with parity zero are also called *even*, while those with parity one are called *odd*.

The symbol α is called *virtual (or auxiliary) symbol*.

In analogy with the ordinary setting that we have discussed in the previous section, we define the *supersymmetric algebra*:

$$\mathbb{C}[M_{1|n,d}] := \mathbb{C}[(\alpha|j), (x_i|j)]_{i=1,\ldots,n;j=1,\ldots,d}$$

as the algebra generated by the $\mathbb{Z}_2$-graded variables $(\alpha|j), (x_i|j)$ subject to the commutation relations:

$$(x_h|i)(x_k|j) = (x_k|j)(x_h|i), \ (x_h|i)(\alpha|j) = (\alpha|j)(x_h|i), \ (\alpha|i)(\alpha|j) = -(\alpha|j)(\alpha|i).$$

Strictly speaking we have:

$$\mathbb{C}[M_{1|n,d}] \cong \bigwedge [(\alpha|j)] \bigotimes \mathrm{Sym}\,[x_{hj}],$$

and, therefore, $\mathbb{C}[M_{1|n,d}]$ is a $\mathbb{Z}_2$−graded algebra, whose $\mathbb{Z}_2$−graduation is inherited by the natural one in the exterior algebra.

Given $a, b \in \{x_1, \ldots, x_n, \alpha\}$, we can define, similarly to what we did before, the operator $D_{a,b}$ which is a *superpolarization*:

$$D_{a,b}\,((c|j)) = \delta_{b,c}\,(a|j), \qquad \forall j = 1, \ldots, d.$$

$x_2 = (x_{21}, x_{22})$ two copies (variabili ausiliarie in Capelli's language) $x_1^{'} = (x_{11}^{'}, x_{12}^{'})$ and $x_2^{'} = (x_{21}^{'}, x_{22}^{'})$, and embed the polynomial algebra $\mathbb{C}[M_{2,2}] = \mathbb{C}[(x_i|j)]_{i,j=1,2}$ we are concerned with into the enlarged polynomial algebra $\mathbb{C}[(x_i|j), (x_i^{'}|j)]_{i,j=1,2}$ (these are the operators that Weyl denotes by the symbol Δ_{x_i, x_j}, and whose composition Weyl called "pseudo-composition"). The polarization operators $D_{x_i^{'}, x_j}$ COMMUTES, and, therefore we may consider the ordinary determinant $det[D_{x_i^{'}, x_j}]$. When we apply the operator

$$H_2^{'} = D_{x_1, x_1^{'}} D_{x_2, x_2^{'}} \ det[D_{x_i^{'}, x_j}]$$

to an algebraic form $f \in \mathbb{C}[M_{2,2}]$, we get the form $[x_1, x_2]\,\Omega_2(f)$. By applying the commutator identities in the enlarged Weyl algebra $\mathbf{W}_{4,2}$, we find that the column determinant $H_{2,2} = \mathbf{cdet}\left(\,D_{x_i, x_j} + \delta_{ij}(2-i)\mathbf{I}\,\right)_{i,j=1,2}$ satisfies the identity $H_{2,2} = H_2^{'} + T$, where T is an operator that acts trivially on the proper algebra $\mathbb{C}[M_{2,2}] = \mathbb{C}[(x_i|j)]_{i,j=1,2}$. In plain words, we can subsitute the action of the column determinant $H_{2,2}$ - which lives in a noncommutative algebra - with the action of the true determinant (in commuting operators) $H_2^{'}$.

In general, $D_{a,b}$ is not a derivation, but a *superderivation* (see, e.g [18], [28]); in other words, we need to introduce a sign in the Leibniz rule if we want to obtain a consistent definition of the operators $D_{a,b}$:

$$D_{a,b}(FG) = D_{a,b}(F)G + (-1)^{|D_{a,b}||F|}FD_{a,b}(G)$$

where: $|F| \in \mathbb{Z}_2$ denotes the $\mathbb{Z}_2-$degree (or, *parity*) of an $\mathbb{Z}_2-$homogeneous element $F \in \mathbb{C}[M_{1|n,d}]$ and

$$|D_{x_h,x_k}| = |D_{\alpha,\alpha}| = 0 \in \mathbb{Z}_2, \qquad |D_{\alpha,x_k}| = |D_{x_h,\alpha}| = 1 \in \mathbb{Z}_2.$$

The space spanned by the superpolarizations form a Lie (sub)superalgebra of the Lie superalgebra $\mathrm{End}(\mathbb{C}[M_{1|n}])$ (see, e.g. [18], [28], [22]), where the Lie (super)bracket is defined introducing also a sign:

$$[A, B] = AB - (-1)^{|A||B|}BA$$

This is called the *supercommutator*.

Claim 3.1. *Given $a, b, c, d \in \{x_1, \ldots, x_n, \alpha\}$, we have*

$$[D_{a,b}, D_{c,d}] = \delta_{b,c}D_{a,d} + (-1)^{|D_{a,b}||D_{c,d}|}\delta_{a,d}D_{c,b}.$$

There is a fundamental yet simple fact coming from a straightforward calculation ($[x_1, \ldots, x_n]$ denotes the *bracket*, eq. (1)):

Lemma 3.2.

$$D_{x_n,\alpha}D_{x_{n-1},\alpha}\cdots D_{x_1,\alpha}\left((\alpha|1)\ldots(\alpha|n-1)(\alpha|n)\right) = [x_1, \ldots, x_n]. \qquad (2)$$

Consider now the *product* of superpolarizations:

$$\mathcal{H}_n = D_{x_n,\alpha}\ldots D_{x_1,\alpha}D_{\alpha,x_1}\ldots D_{\alpha,x_n} \in \mathrm{End}(\mathbb{C}[M_{1|n}]).$$

Claim 3.3. *The subalgebra $\mathbb{C}[M_{n,d}] \hookrightarrow \mathbb{C}[M_{1|n,d}]$ is invariant for the operator $\mathcal{H}_n$.*

Denote by $\mathcal{H}_n \mid_{\mathbb{C}[M_{n,d}]}$ the restriction of the operator $\mathcal{H}_n$ to the subalgebra $\mathbb{C}[M_{n,d}]$.

Claim 3.4. *(see, e.g. [5], [3])*
The restriction $\mathcal{H}_n \mid_{\mathbb{C}[M_{n,d}]}$ can be expressed as an operator in the polarizations D_{x_h,x_k}, $h, k = 1, \ldots, n$. In plain words, it belongs to the classical algebra of polarizations $\boldsymbol{P}_{n,d} \hookrightarrow \boldsymbol{W}_{n,d}$.

$\mathcal{H}_n$ is written as a monomial operator, but its restriction $\mathcal{H}_n \mid_{\mathbb{C}[M_{n,d}]}$ becomes quite different when expressed in terms of classical polarizations. This fact has many consequences in terms of computations, but let us first understand the close connection between $\mathcal{H}_n$ and the classical Capelli operator.

The next result is a special case of the result we called the "Laplace expansion for Capelli rows" ([8] Theorem 2, [3] Theorem 6.3).

Theorem 3.5. [5]

For all $d \in \mathbb{Z}^+$, we have:

$$\mathcal{H}_n \mid_{\mathbb{C}[M_{n,d}]} = H_{n,d}.$$

The proof of Capelli's identities reduces to a straightforward computation. Let us first consider the action of the *virtual* part of $\mathcal{H}_n|_{\mathbb{C}[M_{n,d}]}$, namely:

$$D_{\alpha,x_1} \cdots D_{\alpha,x_{n-1}} D_{\alpha,x_n}$$

on the generic monomial:

$$\mathbf{m} = \prod_{i=1}^{n} \left(\prod_{j=1}^{d} (x_i|j)^{d_{ij}} \right) \in \mathbb{C}[M_{n,d}], \qquad d_{ij} \in \mathbb{N}.$$

Notice that every monomial in the expression

$$D_{\alpha,x_1} \cdots D_{\alpha,x_{n-1}} D_{\alpha,x_n}(\mathbf{m}) \tag{3}$$

contains exactly n occurrences of the anticommuting variables of type:

$$(\alpha|j), \qquad j = 1, \ldots, d.$$

Case $n > d$. To prove the first identity it is enough to observe that all the monomials resulting in the expression (3) contain the square of an odd variable, hence they are all zero. So

$$\mathcal{H}_n(\mathbf{m}) = H_{n,d}(\mathbf{m}) = 0.$$

Hence we have the first identity, by linearity.

Case $n = d$. Let us set $(x_i|j)^{d_{ij}-1} = 0$ if $d_{ij} = 0$, and recall $(\alpha|j)^2 = 0$ for $j = 1, 2, \ldots, d$. We have:

$$D_{\alpha,x_1} \cdots D_{\alpha,x_{n-1}} D_{\alpha,x_n}(\mathbf{m}) =$$

$$= D_{\alpha,x_1} \cdots D_{\alpha,x_{n-1}} D_{\alpha,x_n} \left(\prod_{i=1}^{n} \left(\prod_{j=1}^{d} (x_i|j)^{d_{ij}} \right) \right)$$

$$= \sum_{\sigma \in \mathcal{S}_n} \left(\prod_{i,j=1}^{n} d_{i,\sigma_i}(x_i|j)^{d_{ij}-\delta_{j,\sigma_i}} (\alpha|\sigma_1) \cdots (\alpha|\sigma_{n-1})(\alpha|\sigma_n) \right) =$$

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[5] A sketchy proof of this result can also be found in [23]. From a representation theoretic point of view, Theorem 3.5 is a special case of results that are discussed in the final section of the present paper, to which we refer the reader.

$$= \left(\sum_{\sigma \in \mathcal{S}_n} (-1)^{|\sigma|} \prod_{i,j=1}^{n} d_{i,\sigma_i}(x_i|j)^{d_{ij}-\delta_{j,\sigma_i}} \right) \cdot (\alpha|1) \cdots (\alpha|n-1)(\alpha|n)$$

$$= \Omega_n(\mathbf{m}) \cdot (\alpha|1) \ldots (\alpha|n-1)(\alpha|n)$$

$$= (\alpha|1) \ldots (\alpha|n-1)(\alpha|n) \cdot \Omega_n(\mathbf{m}).$$

We proved:

$$D_{\alpha,x_1} \cdots D_{\alpha,x_n}(\mathbf{m}) = (\alpha|1) \ldots (\alpha|n) \cdot \Omega_n(\mathbf{m}).$$

Applying the operator

$$D_{x_n,\alpha} D_{x_{n-1},\alpha} \cdots D_{x_1,\alpha}$$

to both sides we have:

$$H_{n,n}(\mathbf{m}) = \mathcal{H}_n(\mathbf{m}) = D_{x_n,\alpha} \cdots D_{x_1,\alpha} \cdot D_{\alpha,x_1} \cdots D_{\alpha,x_n}(\mathbf{m})$$

$$= D_{x_n,\alpha} D_{x_{n-1},\alpha} \cdots D_{x_1,\alpha} \left(\; (\alpha|1) \ldots (\alpha|n-1)(\alpha|n) \cdot \Omega_n(\mathbf{m}) \; \right)$$

$$= \left(D_{x_n,\alpha} D_{x_{n-1},\alpha} \cdots D_{x_1,\alpha} \left(\; (\alpha|1) \ldots (\alpha|n-1)(\alpha|n) \; \right) \right) \cdot \Omega_n(\mathbf{m})$$

$$= [x_1, \ldots, x_{n-1}, x_n] \cdot \Omega_n(\mathbf{m}),$$

Hence by linearity we obtain the special identity. $\square$

4 The method of virtual supersymmetric variables for $\mathrm{U}(gl(n))$

Let us consider the vector spaces V_n and the *auxiliary* vector spaces V_{m_1} and V_{m_2} (informally, we assume that $dim(V_{m_1}) = m_1$ and $dim(V_{m_2}) = m_2$ are "sufficiently large"). V_n is called the space of *proper vectors*, and the spaces V_{m_1} and V_{m_2} are called the spaces of *even virtual vectors* and of *odd virtual vectors*, respectively.

Let $W = W_0 \oplus W_1$ be the $\mathbb{Z}_2-$graded vector space, where

$$W_0 = V_{m_1}, \qquad W_1 = V_{m_2} \oplus V_n$$

and let $gl(m_1|m_2+n)$ denote the general linear Lie superalgebra of $W = W_0 \oplus W_1$ (see, e.g. [22], [28], [18]).

Let $A_0 = \{\alpha_1, \ldots, \alpha_{m_1}\}$, $A_1 = \{\beta_1, \ldots, \beta_{m_2}\}$, $L = \{x_1, \ldots, x_n\}$ denote distinguished bases of V_{m_1}, V_{m_2} and V_n, respectively; therefore $|\alpha_s| = 0 \in \mathbb{Z}_2$, and $|\beta_t| = |x_i| = 1 \in \mathbb{Z}_2$.

Let

$$\{e_{a,b}; a, b \in A_0 \cup A_1 \cup L\}, \qquad |e_{a,b}| = |a| + |b| \in \mathbb{Z}_2$$

be the standard $\mathbb{Z}_2$−homogeneous basis of $gl(m_1|m_2 + n)$ provided by the elementary matrices.

The supercommutator of $gl(m_1|m_2 + n)$ has the following explicit form:

$$[e_{a,b}, e_{c,d}] = \delta_{bc}\ e_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}\ e_{c,b},$$

$a, b, c, d \in A_0 \cup A_1 \cup L$.

In analogy with the ordinary setting we discussed in the previous section, the *supersymmetric algebra*

$$\mathbb{C}[M_{m_1|m_2+n,d}]$$

is the algebra generated by the ($\mathbb{Z}_2$-graded) variables $(\alpha_s|j), (\beta_t|j), (x_i|j)$, where

$$|(\alpha_s|j)| = 1 \in \mathbb{Z}_2 \quad and \quad |(\beta_t|j)| = |(x_i|j)| = 0 \in \mathbb{Z}_2,$$

subject to the commutation relations:

$$(a|h)(b|k) = (-1)^{|(a|h)||(b|k)|}\ (b|k)(a|h),$$

for $a, b \in \{\alpha_1, \ldots, \alpha_{m_1}\} \cup \{\beta_1, \ldots, \beta_{m_2}\} \cup \{x_1, x_2, \ldots, x_n\}$.

We have:

$$\mathbb{C}[M_{m_1|m_2+n,d}] \cong \bigwedge [(\alpha_s|j)] \bigotimes \mathrm{Sym}\left[(\beta_t|j), (x_h|j)\right],$$

and, therefore, $\mathbb{C}[M_{m_1|m_2+n,d}]$ is a $\mathbb{Z}_2$−graded algebra (superalgebra), whose $\mathbb{Z}_2$−graduation is inherited by the natural one in the exterior algebra.

Given two symbols $a, b \in A_0 \cup A_1 \cup L$, the *superpolarization* $D_{a,b}$ of b to a is the unique superderivation (see, e.g. [18], [28], [22]) of $\mathbb{C}[M_{m_1|m_2+n,d}]$ of parity $|D_{a,b}| = |a| + |b| \in \mathbb{Z}_2$ such that

$$D_{a,b}\left((c|j)\right) = \delta_{bc}\ (a|j),\ c \in A_0 \cup A_1 \cup L,\ j = 1, \ldots, d. \qquad (4)$$

We have a representation of $gl(m_1|m_2+n)$ on $\mathbb{C}[\mathbb{C}[M_{m_1|m_2+n,d}]]$ sending the elementary matrices in the corresponding superpolarizations:

$$\varrho : gl(m_1|m_2 + n) \to End_{\mathbb{C}}[\mathbb{C}[M_{m_1|m_2+n,d}]]$$

$$e_{a,b} \to D_{a,b}, \qquad a, b \in A_0 \cup A_1 \cup L.$$

Hence this defines a morphism (i.e. a representation):

$$\varrho : \mathbf{U}(gl(m_1|m_2 + n)) \to End_{\mathbb{C}}[\mathbb{C}[M_{m_1|m_2+n,d}]].$$

Definition 4.1. (Irregular expressions and the ideal Irr)

We say that a product

$$e_{a_m b_m} \cdots e_{a_1 b_1} \in \mathbf{U}(gl(m_1|m_2 + n))$$

is an irregular expression whenever the following condition on the occurrences of the virtual variables α_s and β_t holds: there exists a right subsequence $e_{a_i, b_i} \cdots e_{a_2, b_2} e_{a_1, b_1}$, $i \leq m$ and a virtual symbol $\gamma \in A_0 \cup A_1$ such that

$$\#\{j; b_j = \gamma, j \leq i\} > \#\{j; a_j = \gamma, j < i\}. \tag{5}$$

We define the left ideal **Irr** *of* $\mathbf{U}(gl(m_1|m_2 + n))$ *as the left ideal generated by the set of irregular expressions.*

Remark 4.2. The action of any element of **Irr** on the subalgebra $\mathbb{C}[M_{n,d}] \subset \mathbb{C}[M_{m_1|m_2+n,d}]$ - via the representation ϱ - is identically zero.

We have the immersions

$$\mathbf{U}(gl(n)) \equiv \mathbf{U}(gl(0|n)) \hookrightarrow \mathbf{U}(gl(m_1|m_2 + n)),$$

induced by the immersion $gl(n) \hookrightarrow gl(m_1|m_2 + n)$.

Lemma 4.3. *The sum* $\mathbf{U}(gl(0|n)) + $ **Irr** *is direct sum of vector spaces.*

The vector space $Virt(m_1 + m_2, n) = \mathbf{U}(gl(0|n)) \oplus$ **Irr** is a subalgebra of $\mathbf{U}(gl(m_1|m_2 + n))$, which we called the *virtual subalgebra* [6].

Lemma 4.4. **Irr** *is a two sided ideal of* $Virt(m_1 + m_2, n)$.

4.1 The virtual algebra $Virt(m_1 + m_2, n)$ and the virtual presentations of elements in $\mathbf{U}(gl(n))$

Theorem 4.5. (The Capelli epimorphism π)

1. *Every operator in $\varrho[Virt(m_1 + m_2, n)]$ leaves invariant the algebra of algebraic forms $\mathbb{C}[M_{n,d}] \subseteq \mathbb{C}[M_{m_1|m_2+n,d}]$.*

2. *Let $n \leq d$. The morphism*

$$Virt(m_1 + m_2, n) \to \mathrm{End}_{\mathbb{C}}[\mathbb{C}[M_{m_1|m_2+n,d}]] \to \mathrm{End}_{\mathbb{C}}[\mathbb{C}[M_{n,d}]]$$

defines a surjective morphism

$$\pi : Virt(m_1 + m_2, n) \to \mathbf{P}_{n,d} \cong \mathbf{U}(gl(n)), \qquad Ker(\pi) = \textbf{\textit{Irr}}.$$

The "devirtualization" projection operator π is called the *Capelli epimorphism*.

Remark 4.6. Any element in $Virt(m_1+m_2, n)$ defines an element in $\mathbf{U}(gl(n))$, and is called a *virtual presentation* of it. The map π being a surjection, any element $\mathbf{p} \in \mathbf{U}(gl(n))$ admits several virtual presentations.

From the point of view of identities, the idea of the *method of virtual variables* may be informally summarized as follows: in order to prove identities in $\mathbf{U}(gl(n))$

- look for "simple" virtual presentations of the elements involved in;

- prove the identity among the virtual presentations in $Virt(m_1 + m_2, n)$ *modulo* the ideal **Irr**.

Remark 4.7. for every $e_{x_i, x_j} \in gl(n) \subset gl(m_1|m_2+n)$, let $ad(e_{x_i, x_j})$ denote its adjoint action on $Virt(m_1 + m_2, n)$; the ideal **Irr** is $ad(e_{x_i, x_j})$−invariant. Then

$$\pi\left(ad(e_{x_i, x_j})(\mathbf{m})\right) = ad(e_{x_i, x_j})\left(\pi(\mathbf{m})\right), \qquad \mathbf{m} \in Virt(m_1 + m_2, n). \quad (6)$$

Definition 4.8. (Balanced monomials)
In the enveloping algebra $\mathbf{U}(gl(m_1|m_2+n))$, *consider an element of the form:*

$$e_{x_{i_1}, \gamma_{p_1}} \cdots e_{x_{i_n}, \gamma_{p_n}} \cdot e_{\gamma_{p_1}, x_{j_1}} \cdots e_{\gamma_{p_n}, x_{j_n}}, \qquad \gamma_{p_k} \in A_0 \cup A_1, \quad (7)$$

$$(x_{i_1}, \ldots, x_{i_n}, x_{j_1}, \ldots, x_{j_n} \in L, \ i.e. proper \ symbols)$$

that is an element that creates some virtual symbols $\gamma_{p_1}, \ldots, \gamma_{p_n}$ *(with prescribed multiplicities) times an element that annihilates the same virtual symbols (with the same prescribed multiplicities).*

We call such a monomial a balanced monomial.

Proposition 4.9. [6] *Every balanced monomial belongs to* $Virt(m_1 + m_2, n)$. *Hence*

$$\mathbf{p} = \pi\left[e_{x_{i_1}, \gamma_{p_1}} \cdots e_{x_{i_n}, \gamma_{p_n}} \cdot e_{\gamma_{p_1}, x_{j_1}} \cdots e_{\gamma_{p_n}, x_{j_n}}\right] \in \boldsymbol{U}(gl(n)),$$

Furthermore, $\mathbf{p} \in \boldsymbol{U}(gl(n))$ *is independent of the choice of the virtual symbols* γ_{p_h}.

We will say that the balanced monomial

$$e_{x_{i_1}, \gamma_{p_1}} \cdots e_{x_{i_n}, \gamma_{p_n}} \cdot e_{\gamma_{p_1}, x_{j_1}} \cdots e_{\gamma_{p_n}, x_{j_n}}$$

is a *monomial virtual presentation* of the element $\mathbf{p} \in \mathbf{U}(gl(n))$.

The following result lies deeper and is a major tool in the proof of identities involving monomial virtual presentation of elements of $\mathbf{U}(gl(n))$. Since the adjoint representation acts by superderivation, it may be regarded as a version of the *Laplace expansion* for the images of balanced monomials.

---

[6]This result is the (superalgebraic) formalization of the argument developed by Capelli in [14], CAPITOLO I, §X.Metodo delle variabili ausiliarie, page 55 ff.

Proposition 4.10. (Monomial virtual presentation and adjoint actions)
In $\mathbf{U}(gl(n))$, the element

$$\pi\left[e_{x_{i_1},\gamma_{p_1}}\cdots e_{x_{i_n},\gamma_{p_n}}\cdot e_{\gamma_{p_1},x_{j_1}}\cdots e_{\gamma_{p_n},x_{j_n}}\right]$$

equals

$$\pi\left[ad(e_{x_{i_1},\gamma_{p_1}})\cdots ad(e_{x_{i_n},\gamma_{p_n}})\left(e_{\gamma_{p_1},x_{j_1}}\cdots e_{\gamma_{p_n},x_{j_n}}\right)\right].$$

Example 4.11. Let $\alpha \in A_1$. The element

$$\mathbf{p}=\pi\left[e_{x_3,\alpha}e_{x_2,\alpha}e_{x_1,\alpha}\cdot e_{\alpha,x_1}e_{\alpha,x_2}e_{\alpha,x_3}\right]=$$

$$=\pi\left[ad(e_{x_3,\alpha})ad(e_{x_2,\alpha})ad(e_{x_1,\alpha})\left(e_{\alpha,x_1}e_{\alpha,x_2}e_{\alpha,x_3}\right)\right]$$

equals the *column permanent*[7]

$$\mathbf{cper}\begin{pmatrix} e_{x_1,x_1}-2 & e_{x_1,x_2} & e_{x_1,x_3} \\ e_{x_2,x_1} & e_{x_2,x_2}-1 & e_{x_2,x_3} \\ e_{x_3,x_1} & e_{x_3,x_2} & e_{x_3,x_3} \end{pmatrix}\in\mathbf{U}(gl(3)).$$

5 Capelli bitableaux and the Koszul map

Let S and T be two Young tableaux of same shape $\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_p)$ on the same alphabet $L=\{x_1,\ldots,x_n\}$:

$$S=\begin{pmatrix} x_{i_1}\cdots\cdots\cdots & x_{i_{\lambda_1}} \\ x_{j_1}\cdots\cdots x_{j_{\lambda_2}} & \\ \cdots\cdots & \\ x_{s_1}\ldots x_{s_{\lambda_p}} & \end{pmatrix},\qquad T=\begin{pmatrix} x_{h_1}\cdots\cdots\cdots & x_{h_{\lambda_1}} \\ x_{k_1}\cdots\cdots x_{k_{\lambda_2}} & \\ \cdots\cdots & \\ x_{t_1}\ldots x_{t_{\lambda_p}} & \end{pmatrix}\qquad(8)$$

and let $\gamma_1,\ldots,\gamma_p$ be p different virtual symbols of the *same parity*.

In $\mathbf{U}(gl(m_1|m_2+n))$, we consider the monomials:

$$e_{S,\gamma}=\begin{matrix} e_{x_{i_1},\gamma_1}\cdots\cdots\cdots & e_{x_{i_{\lambda_1}},\gamma_1}\cdot \\ e_{x_{j_1},\gamma_2}\cdots\cdots e_{x_{j_{\lambda_2}},\gamma_2}\cdot & \\ \cdots\cdots & \cdot \\ e_{x_{s_1},\gamma_p}\cdots e_{x_{s_{\lambda_p}},\gamma_p} & \end{matrix}$$

and

$$e_{\gamma,T}=\begin{matrix} e_{\gamma_1,x_{h_1}}\cdots\cdots\cdots & e_{\gamma_1,x_{h_{\lambda_1}}}\cdot \\ e_{\gamma_2,x_{k_1}}\cdots\cdots e_{\gamma_2,x_{k_{\lambda_2}}}\cdot & \\ \cdots\cdots & \cdot\qquad\qquad \cdot \\ e_{\gamma_p,x_{t_1}}\cdots e_{\gamma_p,x_{t_{\lambda_p}}} & \end{matrix}$$

and we define the balanced monomial:

$$e_{S,\gamma,T}=e_{S,\gamma}\cdot e_{\gamma,T}\in Virt(m_1+m_2,n)\subseteq\mathbf{U}(gl(m_1|m_2+n)).\qquad(9)$$

---

[7]The symbol $\mathbf{cper}$ denotes the column permanent of a matrix $A=[a_{ij}]$ with noncommutative entries: $\mathbf{cper}(A)=\sum_\sigma a_{\sigma(1),1}a_{\sigma(2),2}\cdots a_{\sigma(n),n}$.

Definition 5.1. (Determinantal and permanental Capelli bitableaux)
  If $\gamma_1, \ldots, \gamma_p \in A_0$, the element

$$[S|T] = \pi\left(e_{S,\gamma,T}\right) \in \mathbf{U}(gl(n))$$

is called a determinantal Capelli bitableau.
  If $\gamma_1, \ldots, \gamma_p \in A_1$, the element

$$[S|T]^* = \pi\left(e_{S,\gamma,T}\right) \in \mathbf{U}(gl(n))$$

is called a permanental Capelli bitableau.
  The balanced monomials $e_{S,\gamma,T}$, $\gamma \in A_0$ ($\gamma \in A_1$) are monomial virtual presentations of the determinantal Capelli bitableau $[S|T]$ (permanental Capelli bitableau $[S|T]^*$).

  By referring to their monomial virtual presentations, we see that determinantal (permanental) Capelli bitableaux are *skew-symmetric* (*symmetric*) with respect to permutations of elements in the same row, both in the tableaux S and T.

Example 5.2. Let $\alpha \in A_0$. Then

$$[x_{i_k} \cdots x_{i_2} x_{i_1} | x_{i_1} x_{i_2} \cdots x_{i_k}] = \pi\left(e_{x_{i_k},\alpha} \cdots e_{x_{i_2},\alpha} e_{x_{i_1},\alpha} \cdot e_{\alpha,x_{i_1}} e_{\alpha,x_{i_2}} \cdots e_{\alpha,x_{i_k}}\right) =$$

$$= \mathbf{cdet} \begin{pmatrix} e_{x_{i_1},x_{i_1}} + (k-1) & e_{x_{i_1},x_{i_2}} & \cdots & e_{x_{i_1},x_{i_k}} \\ e_{x_{i_2},x_{i_1}} & e_{x_{i_2},x_{i_2}} + (k-2) & \cdots & e_{x_{i_2},x_{i_k}} \\ \vdots & \vdots & \vdots & \\ e_{x_{i_k},x_{i_1}} & e_{x_{i_k},x_{i_2}} & \cdots & e_{x_{i_k},x_{i_k}} \end{pmatrix} \in \mathbf{U}(gl(n)).$$

Example 5.3. Let $\alpha, \beta \in A_0$. Then

$$\begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 \end{bmatrix} = \pi\left(e_{x_1,\alpha} e_{x_2,\beta} \cdot e_{\alpha,x_2} e_{\beta,x_1}\right) =$$

$$= -e_{x_1,x_2} e_{x_2,x_1} + e_{x_1,x_1} \in \mathbf{U}(gl(2)).$$

5.1   The Koszul map and the bitableaux isomorphism

Let $\mathbb{C}[(x_i|x_j)]_{i,j=1,\ldots,n}$ denote the polynomial $\mathbb{C}-$algebra in the variables $(x_i|x_j)$, $i, j = 1, \ldots, n$; this algebra is isomorphic to $Sym[gl(n)]$, via the map

$$(x_i|x_j) \mapsto e_{x_i,x_j}, \quad i, j = 1, \ldots, n.$$

  For every x_h, x_k, $h, k = 1, 2, \ldots, n$ let

$$\rho_{x_h,x_k} : \mathbb{C}[(x_i|x_j)]_{i,j=1,\ldots,n} \longrightarrow \mathbb{C}[(x_i|x_j)]_{i,j=1,\ldots,n}$$

be the linear map such that

$$\rho_{x_h,x_k}(\mathbf{M}) = D_{x_h,x_k}(\mathbf{M}) + (x_h|x_k) \cdot \mathbf{M},$$

for every $\mathbf{M} \in \mathbb{C}[(x_i|x_j)]_{i,j=1,...,n}$ (here, the symbol D_{x_h,x_k} denotes the polarization operator

$$D_{x_h,x_k} = \sum_{j=1}^{n} (x_h|x_j) \, \frac{\partial}{\partial(x_k|x_j)}$$

on the algebra $\mathbb{C}[(x_i|x_j)]_{i,j=1,...,n}.$)

The map $e_{x_h,x_k} \to \rho_{x_h,x_k}$ defines a Lie algebra homomorphism

$$T : gl(n) \longrightarrow End_{\mathbb{C}}(\mathbb{C}[(x_i|x_j)]).$$

By the universal property of the enveloping algebra $\mathbf{U}(gl(n))$, the map T uniquely extends to an homomorphism of associative algebras

$$\tau : \mathbf{U}(gl(n)) \longrightarrow (End_{\mathbb{C}}(\mathbb{C}[(x_i|x_j)]), \circ)$$

such that $\tau(e_{x_h,x_k}) = \rho_{x_h,x_k}$ and $\tau(\mathbf{1}) = Id$.

Let now 1 denote the unit of $\mathbb{C}[(x_i|x_j)]_{i,j=1,...,n}$ and let

$$\varepsilon_1 : End_{\mathbb{C}}(\mathbb{C}[(x_i|x_j)]) \longrightarrow \mathbb{C}[(x_i|x_j)]$$

be $\mathbb{C}-$linear map *evaluation at 1*, that is $\varepsilon_1(\rho) = \rho(1)$, for every $\rho \in End_{\mathbb{C}}(\mathbb{C}[(x_i|x_j)])$.

The composite map

$$\mathfrak{K} = \varepsilon_1 \circ \tau : \mathbf{U}(gl(n)) \longrightarrow \mathbb{C}[(x_i|x_j)] \cong Sym[gl(n)]$$

is the *Koszul operator* [23].

We recall that, given a pair (S,T) of the same shape, its *biditerminant* $(S|T)$ is an element of $\mathbb{C}[(x_i|x_j)]$.[8]

Theorem 5.4. *(The standard basis theorem, see e.g. [15], [16], [17])*
The set of biditerminants

$$\{(S|T); \ S, T \, semistandard^9\}$$

is a linear basis of $\mathbb{C}[(x_i|x_j)]$.

We now just state a series of results, which come as an application of the virtual variable method, and refer the reader to [6] and [7] for further details.

---

[8]In this note we assume the "signed" definition of [19], which differs from those of [15], [16], [17] just for a sign. By referring to the notation of eq. (8),

$$(S|T) = \theta \cdot det[(x_{i_{\lambda_1-u+1}}|x_{h_{u'}})]_{u,u'=1,...,\lambda_1} \cdots det[(x_{s_{\lambda_p-v+1}}|x_{t_{v'}})]_{v,v'=1,...,\lambda_p} \in \mathbb{C}[(x_i|x_j)],$$

where $\theta = (-1)^{\lambda_1(\lambda_2+\cdots+\lambda_p)+\cdots+\lambda_{p-1}\lambda_p}$. For a simple presentation of the biditerminant and of its supersymmetric analogues in terms of virtual variables, we refer the reader to [4].

[9]A Young tableau is said to be semistandard if its rows are left to right increasing sequences and its columns are top to bottom nondecreasing sequences.

Theorem 5.5. *([6], [7])*

The map[10]

$$\mathfrak{T} : (S|T) \mapsto [S|T]$$

defines a linear invertible operator

$$\mathbb{C}[(x_i|x_j)]_{i,j=1,\ldots,n} \to \mathbf{U}(gl(n)),$$

whose inverse is the the Koszul operator $\mathfrak{K}$.

Therefore, the Koszul operator $\mathfrak{K}$ is invertible.

Corollary 5.6. *(Koszul, [23])*

$$\mathfrak{K}\left([x_n \ldots x_2 x_1 | x_1 x_2 \ldots x_n]\right) = (x_n \ldots x_2 x_1 | x_1 x_2 \ldots x_n),$$

where

$$(x_n \ldots x_2 x_1 | x_1 x_2 \ldots x_n) = det\,[(x_i|x_j)]_{i,j=1,2,\ldots,n}$$

Besides the notion of the bideterminant $(S|T)$ of the pair (S,T), one has its natural *symmetric* counterpart, namely, its *bipermanent* $(S|T)^* \in \mathbb{C}[(x_i|x_j)]$ (see, e.g [4], [3]).

We recall that the set of co-semistandard[11] bipermanents is a linear basis of $\mathbb{C}[(x_i|x_j)]$ (see, e.g. [19], [3]).

Theorem 5.7. *([6], [7])*

We have:

$$\mathfrak{T} : (S|T)^* \mapsto [S|T]^*,$$

for every $(S|T)^* \in \mathbb{C}[(x_i|x_j)]$.

Corollary 5.8. *([6], [7])*

1. *The set of determinantal Capelli bitableaux:*

$$\{[S|T];\ S,T\ semistandard\} \tag{10}$$

   is a linear basis of $\mathbf{U}(gl(n))$.

2. *The set of permanental Capelli bitableaux:*

$$\{[S|T]^*;\ S,T\ co-semistandard\} \tag{11}$$

   is a linear basis of $\mathbf{U}(gl(n))$.

---

[10]This map is called the "bitableuax correspondence" in [7].

[11]A Young tableau is said to be co-semistandard if its rows are left to right nondecreasing sequences and its columns are top to bottom increasing sequences.

6 The center $\mathfrak{Z}(gl(n))$ of $\mathbf{U}(gl(n))$

In order to make the notation lighter, in this section we simply write $1, 2, \ldots, n$ in place of $x_1, x_2, \ldots, x_n$.

Remark 6.1. Throughout this section the role of Remark 4.7 is ubiquitous: in order to prove that an element is central in $\mathbf{U}(gl(n))$, we simply claim that its virtual presentation in $Virt(m_1 + m_2, n)$ is annihilated by the adjoint actions $ad(e_{i,j}), \ e_{i,j} \in gl(n)$.

Example 6.2. The prototypical example is that of the Capelli element

$$[n \ldots 21 | 12 \ldots n] = \mathbf{cdet}\, [e_{i,j} + \delta_{ij}(n - i)]_{i,j=1,\ldots,n} \in \mathbf{U}(gl(n)),$$

and we discuss it in detail.

First, recall that $ad(e_{i,j})(e_{h,\alpha}) = \delta_{jh} e_{i,\alpha}$, $ad(e_{i,j})(e_{\alpha,k}) = -\delta_{ki} e_{\alpha,j}$, for every virtual symbol α, and that $ad(e_{i,j})$ acts as a derivation.

The monomial

$$M = e_{n,\alpha} \cdots e_{2,\alpha} e_{1,\alpha} e_{\alpha,1} e_{\alpha,2} \cdots e_{\alpha,n} \in Virt(m_1 + m_2, n), \quad \alpha \in A_0,$$

is annihilated by $ad(e_{i,j}), \ i \neq j$, by skew-symmetry. Furthermore, $ad(e_{i,i})(M) = M - M = 0, \ i = 1, 2, \ldots, n$.

Since $[n \ldots 21 | 12 \ldots n] = \pi\, (e_{n,\alpha} \cdots e_{2,\alpha} e_{1,\alpha} e_{\alpha,1} e_{\alpha,2} \cdots e_{\alpha,n}), \quad \alpha \in A_0$, the element $[n \ldots 21 | 12 \ldots n]$ is central in $\mathbf{U}(gl(n))$, by Remark 4.7.

6.1 The classical Capelli generators of 1893

In the enveloping algebra $\mathbf{U}(gl(n))$, given any integer $k = 1, 2, \ldots, n$, consider the element (compare with **Example** 5.2)

$$\mathbf{H}_n^{(k)} = \sum_{1 \leq i_1 < \cdots < i_k \leq n} [i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k] = \tag{12}$$

$$= \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathbf{cdet} \begin{pmatrix} e_{i_1,i_1} + (k-1) & e_{i_1,i_2} & \cdots & e_{i_1,i_k} \\ e_{i_2,i_1} & e_{i_2,i_2} + (k-2) & \cdots & e_{i_2,i_k} \\ \vdots & \vdots & \vdots & \\ e_{i_k,i_1} & e_{i_k,i_2} & \cdots & e_{i_k,i_k} \end{pmatrix}.$$

Notice that the elements $\mathbf{H}_n^{(k)}$ are easily seen to be central, by Remark 6.1.

We recall the following fundamental result, proved by Capelli in two papers ([12], [13]) with deceiving titles (for a faithful description of Capelli's original proof, quite simplified by means of the superalgebraic method of virtual variables, see [7]).

Theorem 6.3. *The set*

$$\mathbf{H}_n^{(1)}, \mathbf{H}_n^{(2)}, \ldots, \mathbf{H}_n^{(n)} = \mathbf{H}_n$$

is a set of algebraically independent generators of the center $\mathfrak{Z}(gl(n))$ of $\mathbf{U}(gl(n))$.

6.2 Rectangular Capelli/Deruyts bitableaux

Given any positive integer p, we define the *rectangular Capelli/Deruyts bitableau*, with p rows:

$$\mathbf{K_n^p} = \begin{bmatrix} n \; n-1 \; \ldots \; 3 \; 2 \; 1 & 1 \; 2 \; 3 \; \ldots \; n-1 \; n \\ n \; n-1 \; \ldots \; 3 \; 2 \; 1 & 1 \; 2 \; 3 \; \ldots \; n-1 \; n \\ \ldots & \ldots \\ & \\ \ldots & \ldots \\ n \; n-1 \; \ldots \; 3 \; 2 \; 1 & 1 \; 2 \; 3 \; \ldots \; n-1 \; n \end{bmatrix} \in \mathbf{U}(gl(n)).$$

From Remark 6.1, we infer:

Proposition 6.4. *The elements $\mathbf{K_n^p}$ are central in $\mathbf{U}(gl(n))$.*

Set, by definition, $\mathbf{K_n^0} = \mathbf{1}$.

Any rectangular Capelli/Deruyts bitableau $\mathbf{K_n^p}$ well behaves on highest weight vectors and therefore, being central, on irreducible representations. The following result directly follows by iterating the first assertion of Proposition 5 of [27].

Proposition 6.5. *(The hook coefficient lemma)*
Let v_μ a highest weight vector of weight $\mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n)$, with $\mu_i \in \mathbb{N}$ for every $i = 1, 2, \ldots, n$. Then

$$\mathbf{K_n^p}(v_\mu) = (-1)^{\binom{p}{2}n} \left(\prod_{i=0}^{p-1} (\mu_1 - i + n - 1)(\mu_2 - i + n - 2) \cdots (\mu_n - i) \right) \cdot v_\mu.$$

The crucial result in this section is that rectangular Capelli/Deruyts bitableaux $\mathbf{K_n^p}$ expand - in a beautiful way - in terms of the classical Capelli generators .

Theorem 6.6. *(Expansion Theorem)* [12]
Let $p \in \mathbb{N}$ and set $\mathbf{H}_n^{(0)} = \mathbf{1}$, by definition. The following identity in $\mathfrak{Z}(gl(n))$ holds:

$$\mathbf{K_n^{p+1}} = (-1)^{np} \; \mathbf{K_n^p} \; \mathbf{C}_n(p), \tag{13}$$

where

$$\mathbf{C}_n(p) = \sum_{j=0}^{n} (-1)^{n-j}(p)_{n-j} \; \mathbf{H}_n^{(j)}, \tag{14}$$

and

$$(p)_k = p(p-1)\cdots(p-k+1), \; p, k \in \mathbb{N}$$

denotes the falling factorial coefficient.

---

[12]This is a new result. It is a special case of a more general result - joint work with F. Regonati and A. Teolis - that will appear in a forthcoming publication.

For $p = 0$, the preceding identity consistently collapses to

$$\mathbf{K_n^1} = \mathbf{H}_n^{(n)} = \mathbf{H}_n = \mathbf{C}_n(0).$$

Furthermore, notice that the linear relations (14), for $p = 0, \ldots, n-1$, yield a nonsingular triangular coefficients matrix.

Corollary 6.7. *The set* $\mathbf{C}_n(0), \mathbf{C}_n(1), \ldots, \mathbf{C}_n(n-1)$ *is a set of algebraically independent generators of the center* $\mathfrak{Z}(gl(n))$ *of* $\mathbf{U}(gl(n))$.

6.3 The Harish-Chandra isomorphism and the algebra $\Lambda^*(n)$ of shifted symmetric polynomials

In this subsection we follow A. Okounkov and G. Olshanski [25].

As in the classical context of the algebra $\Lambda(n)$ of symmetric polynomials in n variables $x_1, x_2, \ldots, x_n$, the algebra $\Lambda^*(n)$ of *shifted symmetric polynomials* is an algebra of polynomials $p(x_1, x_2, \ldots, x_n)$ but the ordinary symmetry is replaced by the *shifted symmetry*:

$$f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i+1} - 1, x_i + 1, \ldots, x_n),$$

for $i = 1, 2, \ldots, n-1$.

Examples 6.8. Two basic classes of shifted symmetric polynomials are provided by the sequences of *shifted elementary symmetric polynomials* and *shifted complete symmetric polynomials*.

- **Shifted elementary symmetric polynomials**

  For every $r \in \mathbb{N}$ let

  $$\mathbf{e}_r^*(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} (x_{i_1} + r - 1)(x_{i_2} + r - 2) \cdots (x_{i_r}), \quad (15)$$

  and $\mathbf{e}_0^*(x_1, x_2, \ldots, x_n) = \mathbf{1}$.

- **Shifted complete symmetric polynomials**

  For every $r \in \mathbb{N}$ let

  $$\mathbf{h}_r^*(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 \leq i_2 < \cdots \leq i_r \leq n} (x_{i_1} - r + 1)(x_{i_2} - r + 2) \cdots (x_{i_r}), \quad (16)$$

  and $\mathbf{h}_0^*(x_1, x_2, \ldots, x_n) = \mathbf{1}$.

The *Harish-Chandra isomorphism* is the algebra isomorphism

$$\chi : \mathfrak{Z}(gl(n)) \longrightarrow \Lambda^*(n), \qquad A \mapsto \chi(A),$$

$\chi(A)$ being the shifted symmetric polynomial such that, for every highest weight module V_μ, the evaluation $\chi(A)(\mu_1, \mu_2, \ldots, \mu_n)$ equals the eigenvalue of $A \in \mathfrak{Z}(gl(n))$ in V_μ ([25], Proposition **2.1**).

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6.4 The Harish-Chandra isomorphism interpretation of Proposition 6.5 and Theorem 6.6

Notice that
$$\chi(\mathbf{H}_n^{(r)}) = \mathbf{e}_r^*(x_1, x_2, \ldots, x_n) \in \Lambda^*(n),$$
for every $r = 1, 2, \ldots, n$.

Furthermore, from Proposition 6.5 it follows

Corollary 6.9. *We have:*

$$\chi(\mathbf{K_n^p}) = (-1)^{\binom{p}{2}n} \left(\prod_{i=0}^{p-1} (x_1 - i + n - 1)(x_2 - i + n - 2) \cdots (x_n - i) \right). \quad (17)$$

By combining equations (13) and (17), we infer:

Proposition 6.10. *We have:*

- *For every $p \in \mathbb{N}$,*

$$\chi(\mathbf{C}_n(p)) = (x_1 - p + n - 1)(x_2 - p + n - 2) \cdots (x_n - p). \quad (18)$$

- *The set*

$$\chi(\mathbf{C}_n(0)), \ \chi(\mathbf{C}_n(1)), \ \ldots \ , \ \chi(\mathbf{C}_n(n-1))$$

is a system of algebraically independent generators of the ring $\Lambda^*(n)$ of shifted symmetric polynomials in the variables $x_1, x_2, \ldots, x_n$.

We recall a standard result (for an elementary proof see e.g. [30]):

Proposition 6.11. *For every $p \in \mathbb{N}$, the element*

$$\mathbf{H}_n(p) = \mathbf{\textit{cdet}} \begin{pmatrix} e_{1,1} - p + (n-1) & e_{1,2} & \cdots & e_{1,n} \\ e_{2,1} & e_{2,2} - p + (n-2) & \cdots & e_{2,n} \\ \vdots & \vdots & \vdots & \\ e_{n,1} & e_{n,2} & \cdots & e_{n,n} - p \end{pmatrix} =$$

$$= \mathbf{\textit{cdet}} \left[e_{i,j} + \delta_{ij}(-p + n - i) \right]_{i,j=1,\ldots,n} \in \mathbf{U}(gl(n)). \quad (19)$$

is central. In symbols, $\mathbf{H}_n(p) \in \mathfrak{Z}(gl(n))$.

Equation (18) implies

$$\chi(\mathbf{H}_n(p)) = (x_1 - p + n - 1)(x_2 - p + n - 2) \cdots (x_n - p) = \chi(\mathbf{C}_n(p)),$$

and, therefore, the following

Corollary 6.12. *For every $p \in \mathbb{N}$, we have*

$$\mathbf{H}_n(p) = \mathbf{C}_n(p) = \sum_{j=0}^{n} (-1)^{n-j}(p)_{n-j} \ \mathbf{H}_n^{(j)}.$$

Let t be a variable and consider the polynomial

$$
\mathbf{H}_n(t) = \mathbf{cdet}
\begin{pmatrix}
e_{1,1} - t + (n-1) & e_{1,2} & \cdots & e_{1,n} \\
e_{2,1} & e_{2,2} - t + (n-2) & \cdots & e_{2,n} \\
\vdots & \vdots & \vdots & \\
e_{n,1} & e_{n,2} & \cdots & e_{n,n} - t
\end{pmatrix} =
$$

$$
= \mathbf{cdet}\left[e_{i,j} + \delta_{ij}(-t + n - i)\right]_{i,j=1,\ldots,n}
$$

with coefficients in $\mathbf{U}(gl(n))$.

Corollary 6.13. *(see, e.g. [30])*
   We have the generating function identity:

$$
\mathbf{H}_n(t) = \sum_{j=0}^{n} (-1)^{n-j}(t)_{n-j}\ \mathbf{H}_n^{(j)},
$$

where, for every $k \in \mathbb{N}$, $(t)_k = t(t-1)\cdots(t-k+1)$ denotes the $k-$th falling factorial polynomial.

Corollary 6.14. *We have the generating function identity:*

$$
\sum_{j=0}^{n} (-1)^{n-j}(t)_{n-j}\ \mathbf{e}_j^*(x_1, x_2, \ldots, x_n) = (x_1 - t + n - 1)(x_2 - t + n - 2)\cdots(x_n - t).
$$

Following Molev [24] Chapt. **7** (see also Brundan [9], Howe and Umeda [21]), consider the "Capelli determinant"

$$
\mathcal{C}_n(s) = \mathbf{cdet}
\begin{pmatrix}
e_{1,1} + s & e_{1,2} & \cdots & e_{1,n} \\
e_{2,1} & e_{2,2} + s - 1 & \cdots & e_{2,n} \\
\vdots & \vdots & \vdots & \\
e_{n,1} & e_{n,2} & \cdots & e_{n,n} + s - (n-1)
\end{pmatrix} =
$$

$$
= \mathbf{cdet}\left[e_{i,j} + \delta_{ij}(s - i + 1)\right]_{i,j=1,\ldots,n},
$$

regarded as a polynomial in the variable s.

By the formal (column) Laplace rule, the coefficients $\mathcal{C}_n^{(h)} \in \mathbf{U}(gl(n))$ in the expansion

$$
\mathcal{C}_n(s) = s^n + \mathcal{C}_n^{(1)} s^{n-1} + \mathcal{C}_n^{(2)} s^{n-2} + \ldots + \mathcal{C}_n^{(n)},
$$

are the sums of the minors:

$$
\mathcal{C}_n^{(h)} = \sum_{1 \leq i_1 < i_2 < \ldots < i_h \leq n} \mathcal{M}_{i_1, i_2, \ldots, i_h},
$$

where $\mathcal{M}_{i_1, i_2, \ldots, i_h}$ denotes the column determinant of the submatrix of the matrix $\mathcal{C}_n(0)$ obtained by selecting the rows and the columns with indices $i_1 < i_2 < \ldots < i_h$.

Since $\mathcal{C}_n(s) = \mathbf{H}_n(-s + (n-1))$, from Proposition 6.13 it follows:

Corollary 6.15.

$$\mathcal{C}_n(s) = \sum_{j=0}^{n} (-1)^{n-j} (-s + (n-1))_{n-j} \, \mathbf{H}_n^{(j)}.$$

Remark 6.16. Recall that the polynomial sequence of *falling factorials* $((x)_n)_{n \in \mathbb{N}}$ is a *polynomial sequence of binomial type* and that one has the basic linear relations

$$(x)_n = \sum_{k=0}^{n} \mathrm{s}(n,k) \, x^k, \qquad n \in \mathbb{N},$$

where the symbol $\mathrm{s}(n,k)$ denotes the *Stirling numbers of the first kind* (see, e.g. [1]).

A straightforward computation leads to the linear relations:

$$\mathcal{C}_n^{(n-h)} = (-1)^h \sum_{j=0}^{n} c(n-h,j) \, (n-j)! \, \mathbf{H}_n^{(j)},$$

where

$$c(n-h,j) = \sum_{p=0}^{n-j} \binom{n-1}{j+p-1} \frac{\mathrm{s}(p,h)}{p!}. \tag{20}$$

Since the coefficients in (20) yield a nonsingular triangular matrix, Remark 6.16 implies:

Corollary 6.17. *We have:*

- *The elements $\mathcal{C}_n^{(h)}$, $h = 1, 2, \ldots, n$ are central and provide a system of algebraically independent generators of $\mathfrak{Z}(gl(n))$.*

-
$$\chi(\mathcal{C}_n^{(h)}) = \bar{\mathbf{e}}_h(x_1, x_2, \ldots, x_n) = \mathbf{e}_h(x_1, x_2 - 1, \ldots, x_n - (n-1)),$$

 where $\mathbf{e}_h$ denotes the $h-th$ elementary symmetric polynomial.

6.5 Permanental generators

We end this section by describing, in the virtual variables notation, the set of the preimages in $\mathfrak{Z}(gl(n))$ - with respect to the Harish-Chandra isomorphism - of the sequence of *shifted complete symmetric polynomials* $\mathbf{h}_r^*(x_1, x_2, \ldots, x_n)$.

Definition 6.18. *For every $r \in \mathbb{Z}^+$, set*

$$\mathcal{P}_n^{(r)} = \sum_{(i_1, i_2, \ldots, i_n)} (i_1! i_2! \cdots i_n!)^{-1} \, [n^{i_n} \cdots 2^{i_2} 1^{i_1} | 1^{i_1} 2^{i_2} \cdots n^{i_n}]^*, \tag{21}$$

where the sum is extended to all $n-$tuples $(i_1, i_2, \ldots, i_n)$ such that $i_1 + i_2 + \cdots + i_n = r$ and any

$$[n^{i_n} \cdots 2^{i_2} 1^{i_1} | 1^{i_1} 2^{i_2} \cdots n^{i_n}]^*$$

is a permanental Capelli bitableau with one row.

The proofs of the following results are almost trivial, as a consequence of the definition of the elements $\mathcal{P}_n^{(r)}$ in terms of their *monomial virtual presentations* (equation (21)).

Theorem 6.19. *We have:*

- *For every $r \in \mathbb{Z}^+$, the element $\mathcal{P}_n^{(r)}$ is central. In symbols, $\mathbf{P}_n^{(r)} \in \mathfrak{Z}(gl(n))$.*

- *For every $r \in \mathbb{Z}^+$, $\chi(\mathcal{P}_n^{(r)}) = \mathbf{h}_r^*(x_1, x_2, \ldots, x_n)$.*

Amazingly, the generators $\mathcal{P}_n^{(r)}$ coincide with the "permanental generators" of $\mathfrak{Z}(gl(n))$ first discovered and studied - through a rather heavy machinery - by Umeda and Hirai [31] (see also Turnbull [32]).

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